

A categorical characterization of relative entropy on Polish spaces

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Abstract

We give a categorical treatment, in the spirit of Baez and Fritz, of relative entropy for probability distributions defined on Polish spaces. We define a category called PolStat suitable for reasoning about statistical inference on Polish spaces. We define relative entropy as a functor into Lawvere's category $[0, \infty]$ and we show convexity, lower semicontinuity and uniqueness.

1 Introduction

The inspiration for the present work comes from two recent developments. The first is the beginning of a categorical understanding of Bayesian inversion and learning [DG15, DDG16, CDDG17] the second is a categorical reconstruction of relative entropy [BFL11, BF14, Lei]. The present paper provides a categorical treatment of entropy in the spirit of Baez and Fritz in the setting of Polish spaces, thus setting the stage to explore the role of entropy in learning.

Recently there have been some exciting developments that bring some categorical insights to probability theory and specifically to learning theory. These are reported in some recent papers by Clerc, Dahlqvist, Danos and Garnier [DG15, DDG16, CDDG17]. The first of these papers showed how to view the Dirichlet distribution as a natural transformation thus opening the way to an understanding of higher-order probabilities, while the second gave a powerful framework for constructing several natural transformations. In [DG15] the hope was expressed that one could use these ideas to understand Bayesian inversion, a core concept in machine learning. In [CDDG17] this was realized in a remarkably novel way. These papers carry out their investigations in the setting of Polish spaces and are based on the Gir monad [Gir81, Law64].

In [BFL11, BF14] a beautiful treatment of relative entropy is given in categorical terms. The basic idea is to understand entropy in terms of the results of experiments and observations. How much does one learn about a probabilistic situation by doing experiments and observing the results? A category is set up where the morphisms capture the interplay between the original space and the space of observations. In order to interpret the relative entropy as a functor they use Lawvere’s category which consists of a single object and a morphism for every extended positive real number [Law73].

Our contribution is to develop the theory of Baez et al. in the setting of Polish spaces; their work is carried out with finite sets. While the work of [BF14] gives a firm conceptual direction, it gives little guidance in the actual development of the mathematical theory. We had to redevelop the mathematical framework and find the right analogues for the concepts appropriate to the finite case. In particular the proofs of lower semi-continuity, convexity and uniqueness were challenging.

2 Background

In this section we review some of the background. We assume that the reader is familiar with concepts from topology and measure theory as well as basic category theory. We have found books by Ash [Ash72], Billingsley [Bil95] and Dudley [Dud89] to be useful. For probability measures on Polish spaces, we have found “Probability Measures on Metric Spaces” by Parthasarathy [Par67] invaluable.

We will use letters like X, Y, Z for measurable spaces or Polish spaces and capital Greek letters like Σ, Λ, Ω for σ -algebras. We will use μ, ν for measures and p, q, \dots for probability measures. Given (X, Σ) and (Y, Λ) and a measurable function $f : X \rightarrow Y$ and a measure μ on (X, Σ) we obtain a measure on (Y, Λ) by $\mu \circ f^{-1}$; this is called the *pushforward* measure or the *image* measure. We sometimes write $f_*(\mu)$ for this measure.

2.1 Classical Theorems

The Radon-Nikodym is the main tool used to show the existence of conditional probability distributions. We say a measure μ is *absolutely continuous* with respect to another measure ν on the same measurable space X , denoted by $\mu \ll \nu$, if for all measurable sets B , $\nu(B) = 0$ implies that $\mu(B) = 0$.

Theorem 1 (Radon-Nikodym). Given a measurable space (X, Σ) if a σ -finite measure μ is absolutely continuous with respect to a σ -finite measure ν on (X, Σ) , then there is a measurable function $f : X \rightarrow [0, \infty)$, such that for any

measurable subset $A \subset X$,

$$\mu(A) = \int_A f \, d\nu.$$

The function f is unique up to a ν -null set and is called the *Radon-Nikodym derivative*, denoted by $\frac{d\mu}{d\nu}$.

We recall the chain rule for the Radon-Nikodym derivative: given probability measures p, q and m on X with $p \ll q \ll m$, then we have

$$\frac{dp}{dm} = \frac{dp}{dq} \frac{dq}{dm}, \text{ } p\text{-almost everywhere.}$$

A crucial construct that we need is the Markov kernel; this is the generalization of the transition probability matrix from the theory of discrete Markov chains.

Definition 2. Given measurable spaces (X, Σ) and (Y, Λ) a *Markov kernel* is a function $h : X \times \Lambda \rightarrow [0, 1]$ such that for each $x \in X, h(x, \cdot) : \Lambda \rightarrow [0, 1]$ is a (sub)probability measure and for each $A \in \Sigma, h(\cdot, A) : X \rightarrow [0, 1]$ is a measurable function with respect to the Borel algebra on $[0, 1]$.

The Radon-Nikodym theorem is very general but it does not give as strong regularity features as one might want. In particular, if we use it to construct Markov kernels we have the following problem. The putative kernel h , may fail to be countably additive as a measure $h(x, \cdot)$, for a particular pairwise disjoint countable family for x in a set of measure 0. However, when one considers all possible pairwise disjoint countable families, h may fail to be countably additive everywhere. A stronger theorem is needed; this is the so-called *disintegration theorem*. It requires stronger hypotheses on the space on which the kernels are being defined. The objects obtained are often called *regular conditional probability distributions*. The following is the main theorem [Ash72, CP97].

Theorem 3 (Disintegration). Let (X, p) and (Y, q) be two Polish spaces equipped with probability measures, where q is the pushforward measure $q := p \circ f^{-1}$ for a Borel measurable function $f : X \rightarrow Y$. Then, there exists a q -almost everywhere uniquely determined family of probability measures $\{p_y\}_{y \in Y}$ such that

- (i) the function $y \mapsto p_y(B)$ is a Borel-measurable function for each Borel-measurable set $B \subset Y$;
- (ii) p_y is supported on the fiber $f^{-1}(y)$: for q -almost all $y \in Y$;
- (iii) for every Borel-measurable function $f : X \rightarrow [0, \infty]$,

$$\int_X f \, dp = \int_Y \int_{f^{-1}(y)} f \, dp_y \, dq.$$

2.2 Polish Spaces and the Giry Monad

The arena for the results described in this paper is the category of Polish spaces \mathbf{Pol} .

Definition 4. A **Polish space** is a separable, completely metrizable topological space. The category **Pol** has Polish spaces as objects and continuous maps as the morphisms.

Note that a Polish space only refers to the topology; the metric which produces the topology is not unique and is not part of the structure. A typical Polish space is \mathbb{R} . Another, less obvious, example is $(0, 1)$, which is a Polish space (being homeomorphic to \mathbb{R}) despite the fact that it is not complete with its “usual” metric. We always use the Borel algebra on a Polish space.

The natural topology on the space of measures of a Polish space is the *weak topology*. If X is a Polish space, we write $\mathcal{G}(X)$ for the space of measures on the Borel algebra of X ; the letter \mathcal{G} is used in honour of Michèle Giry [Gir81]. Given a probability measure p on X , we define a neighbourhood base for p by taking sets of the form:

$$B_{f_1, \dots, f_n; \varepsilon_1, \dots, \varepsilon_n} := \{q : \in \mathcal{G}(X) : |\int f_i dp - \int f_i dq| < \varepsilon_i, \quad i = 1, 2, \dots, n.\}$$

where the f_i are continuous real-valued functions and the ε_i are positive real numbers. This defines the **weak** topology. It topologizes *weak convergence* which means that a net $\{p_\alpha\}$ converges to p iff the net $\int f dp_\alpha$ converges to $\int f dp$ for any bounded real-valued uniformly-continuous functions f . This is very nicely described in Section 6 of [Par67]; see, in particular Theorem 6.1.

If X is a Polish space, then it can be shown that the space of probability measures on the Borel subsets of X endowed with the weak topology is also Polish [Par67, Theorem 6.5].

We are now ready to define the Giry monad; henceforth we will call it the Giry monad.

Definition 5. The functor $\mathcal{G} : \mathbf{Pol} \rightarrow \mathbf{Pol}$ maps a Polish space X to $\mathcal{G}(X)$, the space of probability measures on X endowed with the weak topology. A morphism $f : X \rightarrow Y$ in **Pol** is mapped to $\mathcal{G}(f) : \mathcal{G}(X) \rightarrow \mathcal{G}(Y)$ by $\mathcal{G}(f)(p) = p \circ f^{-1}$. The natural transformation $\eta : I \rightarrow \mathcal{G}$ is given by $\eta_X(x) = \delta_x$, the Dirac measure concentrated at x . The monad multiplication $\mu : \mathcal{G}^2 \rightarrow \mathcal{G}$ is given by

$$\forall A \in \mathcal{B}(X), \quad \mu_X(M)(A) := \int_{\mathcal{G}(X)} ev_A \, dM$$

where M is a measure in $\mathcal{G}(\mathcal{G}(X))$ and $ev_A : \mathcal{G}(X) \rightarrow [0, 1]$ is the measurable function on $\mathcal{G}(X)$ defined by $ev_A(p) = p(A)$.

The natural transformation μ uses M to average over the space of probability distributions.

The Kleisli category of this monad has as objects Polish spaces and as morphisms maps from X to $\mathcal{G}(Y)$: $h : X \rightarrow (\mathcal{B}_Y \rightarrow [0, 1])$ which are continuous. Here \mathcal{B}_Y stands for the Borel sets of Y and $\mathcal{G}(Y)$ has the weak topology. Now we can curry this to write it as $h : X \times \mathcal{B}_Y \rightarrow [0, 1]$ or $h(x, U)$ where x is a point in X and U is a Borel set in Y . Written this way it is called a Markov kernel and

one can view it as a transition probability function or conditional probability distribution given x . Composition of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in the Kleisli category is given by the formula

$$(g \circ f)(x, V \subset Z) = \int_Y g(y, V) df(x, \cdot).$$

In order for the above integral to be well defined, one needs to check that $g(\cdot, V)$ is measurable; this is proved in [Gir81].

For an arrow $s : Y \rightarrow \mathcal{G}(X)$ in **Pol**, we write s_y for $s(y)$ or, in kernel form $s(y, \cdot)$. For arrows $t : Z \rightarrow \mathcal{G}(Y)$ and $s : Y \rightarrow \mathcal{G}(X)$ in **Pol**, we denote their Kleisli composition by $s \tilde{\circ} t := \mu_X \circ \mathcal{G}(s) \circ t$. For Polish spaces equipped with a probability measure p , we sometimes omit the measure in the notation, *i.e.* we sometimes write X instead of (X, p) .

We note that absolute continuity is preserved by Kleisli composition; the proof is straightforward.

Proposition 6. Given a Polish space Y with probability measures q and q' such that $q \ll q'$. Then, for arbitrary Polish space X and morphism s from Y to $\mathcal{G}(X)$, we have $s \tilde{\circ} q \ll s \tilde{\circ} q'$.

3 The categorical setting

In this section, following Baez and Fritz [BF14] (see also [BFL11]) we describe the categories **FinStat** and **FP** which they use for their characterization of entropy on finite spaces. We then introduce the category **PolStat** which will be the arena for the generalization to Polish spaces.

Before doing so, we define the notion of coherence which will play an important role in what follows.

Definition 7. Given Polish spaces X and Y with probability measure p and q , respectively, a pair (f, s) , $f : (X, p) \rightarrow (Y, q)$ and $s : Y \rightarrow \mathcal{G}(X)$, is said to be *coherent* when f is measure preserving, *i.e.*, $f_*(p) = q$, and the support of s_y is contained in $f^{-1}(y)$. If in addition, p is absolutely continuous with respect to $s \tilde{\circ} q$, then we say that (f, s) is *absolutely coherent*.

We identify a finite set as a Polish space by giving it the discrete topology.

Definition 8. The category **FinStat** has

- **Objects** : Pairs (X, p) where X is a finite set and p a probability measure on X .
- **Morphisms** : $Hom(X, Y)$ are all coherent pairs (f, s) , $f : X \rightarrow Y$ and $s : Y \rightarrow \mathcal{G}(X)$.

We compose arrows $(f, s) : (X, p) \rightarrow (Y, q)$ and $(g, t) : (Y, q) \rightarrow (Z, m)$ as follows: $(g, t) \circ (f, s) := (g \circ f, s \tilde{\circ}_{fin} t)$ where $\tilde{\circ}_{fin}$ is defined as

$$(s \tilde{\circ}_{fin} t)_z(x) = \sum_{y \in Y} t_z(y) s_y(x).$$

One can think of f as a measurement process from X to Y and of s as a hypothesis about X given an observation in Y . We say that a hypothesis s is *optimal* if $p = s \tilde{\circ}_{fin} q$. We denote by **FP** the subcategory of **FinStat** consisting of the same objects, but with only those morphisms where the hypothesis is optimal. See [BFL11, BF14] and [Lei] for a discussion of these ideas.

We now leave the finite world for a more general one: the category **Pol**.

Definition 9. The category **PolStat** has

- **Objects** : Pairs (X, p) where X is a Polish space and p a probability measure on the Borel subsets of X .
- **Morphisms** : $Hom(X, Y)$ are all coherent pairs (f, s) , $f : X \rightarrow Y$ and $s : Y \rightarrow \mathcal{G}(X)$.

We compose arrows $(f, s) : (X, p) \rightarrow (Y, q)$ and $(g, t) : (Y, q) \rightarrow (Z, m)$ as follows: $(g, t) \circ (f, s) := (g \circ f, s \tilde{\circ} t)$.

Following the graphical representation from [BF14] we represent composition as follows:

$$\begin{array}{ccccc} (X, p) & \xleftarrow{s} & (Y, q) & \xleftarrow{t} & (Z, m) \\ & \searrow f & & \searrow g & \\ & & & & \end{array} \xRightarrow{\text{Composition}} \begin{array}{ccc} (X, p) & \xleftarrow{s \tilde{\circ} t} & (Z, m) \\ & \searrow g \circ f & \end{array} .$$

Proposition 10. Given coherent pairs the composition is coherent. If, in addition, they are absolutely coherent, the composition is absolutely coherent.

Proof. Let (f, s) , $f : X \rightarrow Y$ and $s : Y \rightarrow \mathcal{G}(X)$, and (g, t) , $g : Y \rightarrow Z$ and $t : Z \rightarrow \mathcal{G}(Y)$, be coherent pairs. We show the pair $(g \circ f, s \tilde{\circ} t)$ is coherent. Since the composition of two measure preserving functions is also measure preserving, $g \circ f$ is measure preserving. It remains to show that, for all $z \in Z$, the support of $(s \tilde{\circ} t)_z$ is contained in $f^{-1}(g^{-1}(z))$. Let $z \in Z$ be arbitrary. Let $A \subset X$ be an arbitrary measurable set disjoint from $f^{-1}(g^{-1}(z))$. We argue that $(s \tilde{\circ} t)_z(A) = 0$. For all $y \in g^{-1}(z)$, we have $f^{-1}(y)$ disjoint from A . Hence, since (f, s) is coherent, for all $y \in g^{-1}(z)$, we have $s_y(A) = 0$. Therefore,

$$(s \tilde{\circ} t)_z(A) = \int_{y \in g^{-1}(z)} s_y(A) dt_z = 0.$$

Next, in addition, assume the pairs (f, s) and (g, t) are absolutely coherent. We show $p \ll (s \tilde{\circ} t \tilde{\circ} m)$. By hypotheses, $p \ll s \tilde{\circ} q$ and $q \ll t \tilde{\circ} m$. Using

proposition 6 on $q \ll t \tilde{\circ} m$, we get $s \tilde{\circ} q \ll s \tilde{\circ} t \tilde{\circ} m$. By transitivity of \ll , we conclude $p \ll (s \tilde{\circ} t \tilde{\circ} m)$. \blacksquare

We end this section by defining one more category; this one is due to Lawvere [Law73]. It is just the set $[0, \infty]$ but endowed with categorical structure. This allows numerical values associated with morphisms to be regarded as functors.

Definition 11. The category $[0, \infty]$ has

- **Objects** : One single object: \bullet .
- **Morphisms** : For each element $r \in [0, \infty]$, one arrow $r : \bullet \rightarrow \bullet$.

Arrow composition is defined as addition in $[0, \infty]$.

This is a remarkable category with monoidal closed structure and many other interesting properties.

4 Relative entropy functor

We recapitulate the definition of the relative entropy functor on **FinStat** from Baez and then extend it to **PolStat**.

Definition 12. The relative entropy functor RE_{fin} is defined on **FinStat** as :

- **Objects** : Maps every object (X, p) to \bullet .
- **Morphisms** : Maps an arrow $(f, s) : (X, p) \rightarrow (Y, q)$ to $S_{fin}(p, s \tilde{\circ}_{fin} q)$, where

$$S_{fin}(p, s \tilde{\circ}_{fin} q) := \sum_{x \in X} p(x) \ln \left(\frac{p(x)}{(s \tilde{\circ}_{fin} q)(x)} \right).$$

The convention from now on will be that $\infty \cdot c = c \cdot \infty = \infty$ for $0 < c \leq \infty$, but $\infty \cdot 0 = 0$. We extend RE_{fin} from **FinStat** to **PolStat**.

Definition 13. The relative entropy functor RE is defined on **PolStat** as :

- **Objects** : Maps every object (X, p) to \bullet .
- **Morphisms** : Maps every absolutely coherent morphism $(f, s) : (X, p) \rightarrow (Y, q)$ to $S(p, s \tilde{\circ} q)$, where

$$S(p, s \tilde{\circ} q) := \int_X \log \left(\frac{dp}{d(s \tilde{\circ} q)} \right) dp, \text{ where } \frac{dp}{d(s \tilde{\circ} q)} \text{ is the Radon-Nikodym derivative}$$

and otherwise maps to ∞ .

This quantity is also known as the *Kullback-Leibler divergence*.

We could have defined our category to have only absolutely coherent morphisms but it would make the comparison with the finite case more awkward. The

present definition leads to slightly awkward proofs where we have to consider absolutely coherent pairs and ordinary coherent pairs separately.

Clearly, RE restricts to RE_{fin} on **FinStat**. If (f, s) is absolutely coherent, then p is absolutely continuous with respect to $(s \circ q)$ and the Radon-Nikodym derivative is defined. The relative entropy is always non-negative [KL51]; this is an easy consequence of Jensen's inequality. This shows that RE is defined everywhere in **PolStat**.

We will use the following notation occasionally:

$$RE \left(\begin{array}{ccc} & \xleftarrow{s} & \\ (X, p) & & (Y, q) \\ & \xrightarrow{f} & \end{array} \right) := RE((f, s))$$

and also

$$RE \left(\begin{array}{ccccc} & \xleftarrow{s} & & \xleftarrow{t} & \\ (X, p) & & (Y, q) & & (Z, m) \\ & \xrightarrow{f} & & \xrightarrow{g} & \end{array} \right) := RE \left(\begin{array}{ccc} & \xleftarrow{s \circ t} & \\ (X, p) & & (Z, m) \\ & \xrightarrow{g \circ f} & \end{array} \right).$$

It remains to show that RE is indeed a functor. That is, we want to show that

$$\begin{aligned} & RE \left(\begin{array}{ccccc} & \xleftarrow{s} & & \xleftarrow{t} & \\ (X, p) & & (Y, q) & & (Z, m) \\ & \xrightarrow{f} & & \xrightarrow{g} & \end{array} \right) \\ &= RE \left(\begin{array}{ccc} & \xleftarrow{s} & \\ (X, p) & & (Y, q) \\ & \xrightarrow{f} & \end{array} \right) + RE \left(\begin{array}{ccc} & \xleftarrow{t} & \\ (Y, q) & & (Z, m) \\ & \xrightarrow{g} & \end{array} \right). \end{aligned}$$

In order to do so, we will need the following lemma.

Lemma 14. The relative entropy is preserved under pre-composition by optimal hypotheses. In other words, we have

$$RE \left(\begin{array}{ccc} & \xleftarrow{t} & \\ (Y, q) & & (Z, m) \\ & \xrightarrow{g} & \end{array} \right) = RE \left(\begin{array}{ccccc} & \xleftarrow{s} & & \xleftarrow{t} & \\ (X, s \circ q) & & (Y, q) & & (Z, m) \\ & \xrightarrow{f} & & \xrightarrow{g} & \end{array} \right)$$

Proof. Case I : (g, t) is absolutely coherent. Since (g, t) is absolutely coherent, so is $(g \circ f, s \circ t)$ by Proposition 6. Hence, to show $RE(g, t) = RE(g \circ f, s \circ t)$ is to show

$$\int_Y \log \left(\frac{dq}{d(t \circ m)} \right) dq = \int_X \log \left(\frac{d(s \circ q)}{d(s \circ t \circ m)} \right) d(s \circ q).$$

Because f is measure preserving, it is sufficient to show that the two following functions on X are

$$\frac{dq}{d(t \tilde{\circ} m)} \circ f = \frac{d(s \tilde{\circ} q)}{d(s \tilde{\circ} t \tilde{\circ} m)} \quad s \tilde{\circ} t \tilde{\circ} m\text{-almost everywhere.}$$

By the Radon-Nikodym theorem, it is sufficient to show that for any $E \subset X$ measurable set, we have

$$(s \tilde{\circ} q)(E) = \int_E \frac{dq}{d(t \tilde{\circ} m)} \circ f \, d(s \tilde{\circ} t \tilde{\circ} m).$$

The following calculation establishes the above.

$$\int_E \frac{dq}{d(t \tilde{\circ} m)} \circ f \, d(s \tilde{\circ} t \tilde{\circ} m) \tag{1}$$

$$= \int_Y \left(\int_{x \in f^{-1}(y) \cap E} \left(\frac{dq}{d(t \tilde{\circ} m)} \circ f \right) (x) \, d(s \tilde{\circ} t \tilde{\circ} m)_y \right) d(t \tilde{\circ} m) \tag{2}$$

$$= \int_Y \frac{dq}{d(t \tilde{\circ} m)}(y) \left(\int_{f^{-1}(y) \cap E} d(s \tilde{\circ} t \tilde{\circ} m)_y \right) d(t \tilde{\circ} m) \tag{3}$$

$$= \int_Y \frac{dq}{d(t \tilde{\circ} m)}(y) \left(\int_{f^{-1}(y) \cap E} ds_y \right) d(t \tilde{\circ} m) \tag{4}$$

$$= \int_Y \frac{dq}{d(t \tilde{\circ} m)}(y) s_y(E \cap f^{-1}(y)) d(t \tilde{\circ} m) \tag{5}$$

$$= \int_Y \frac{dq}{d(t \tilde{\circ} m)}(y) s_y(E) d(t \tilde{\circ} m) \tag{6}$$

$$= \int_Y s_y(E) \, dq \tag{7}$$

$$= (s \tilde{\circ} q)(E) \tag{8}$$

We get (2) by applying the disintegration theorem to $f : (X, s \tilde{\circ} t \tilde{\circ} m) \rightarrow (Y, t \tilde{\circ} m)$. The equation (3) follows by using the fact that $\frac{dq}{d(t \tilde{\circ} m)} \circ f$ is constant on $f^{-1}(y)$ for every y . To obtain (4) we apply Lemma 22. To show (6) we use the fact that the support of s_y is contained in $f^{-1}(y)$. We get (7) by the definition of the Radon-Nikodym and we finally establish (8) by the definition of Kleisli composition.

Case II : (g, t) is not absolutely coherent. We have $RE((g, t)) = \infty$. We show that $(g \circ f, s \tilde{\circ} t)$ is not absolutely coherent, i.e., $s \tilde{\circ} q$ is not absolutely continuous with respect to $s \tilde{\circ} t \tilde{\circ} m$.

Since, by hypothesis, $q \ll t \tilde{\circ} m$ doesn't hold, there exists a measurable set $B \subset Y$ such that $(t \tilde{\circ} m)(B) = 0$ but $q(B) > 0$. We argue that $(s \tilde{\circ} t \tilde{\circ} m)(f^{-1}(B)) =$

0 and $(s \tilde{\circ} q)(f^{-1}(B)) > 0$. On one hand, we have

$$(s \tilde{\circ} t \tilde{\circ} m)(f^{-1}(B)) = \int_B s_y(f^{-1}(B)) d(t \tilde{\circ} m) \leq (t \tilde{\circ} m)(B) = 0.$$

But on the other hand, since f is a measure preserving map from $(X, s \tilde{\circ} q)$ to (Y, q) , we have $(s \tilde{\circ} q)(f^{-1}(B)) = q(B) > 0$.

Therefore,

$$RE((g, t)) = \infty = RE((g \circ f, s \tilde{\circ} t)).$$

■

Theorem 15 (Functoriality). Given arrows $(f, s) : (X, p) \rightarrow (Y, q)$ and $(g, t) : (Y, q) \rightarrow (Z, m)$, we have

$$RE((g, t) \circ (f, s)) = RE((f, s)) + RE((g, t)).$$

Proof. Note that by definition, $RE((g, t) \circ (f, s)) = RE((g \circ f, s \tilde{\circ} t))$.

Case I : (f, s) and (g, t) are absolutely coherent. By proposition 10, we have that $(g \circ f, s \tilde{\circ} t)$ is absolutely coherent.

$$RE((g \circ f, s \tilde{\circ} t)) = \int_X \log \left(\frac{dp}{d(s \tilde{\circ} t \tilde{\circ} m)} \right) dp \quad (9)$$

$$= \int_X \log \left(\frac{dp}{d(s \tilde{\circ} q)} \frac{d(s \tilde{\circ} q)}{d(s \tilde{\circ} t \tilde{\circ} m)} \right) dp \quad (10)$$

$$= \int_X \log \left(\frac{dp}{d(s \tilde{\circ} q)} \right) dp + \int_X \log \left(\frac{d(s \tilde{\circ} q)}{d(s \tilde{\circ} t \tilde{\circ} m)} \right) dp \quad (11)$$

$$= RE((f, s)) + \int_X \log \left(\frac{d(s \tilde{\circ} q)}{d(s \tilde{\circ} t \tilde{\circ} m)} \right) dp \quad (12)$$

$$= RE((f, s)) + RE((g, t)) \quad (13)$$

We get (10) by the chain rule for Radon-Nikodym derivatives and (13) by applying lemma 14.

Case II : (g, t) is not absolutely coherent. We argue that $(g \circ f, s \tilde{\circ} t)$ is not absolutely coherent. By hypothesis, $q \ll t \tilde{\circ} m$ doesn't hold, so there is a measurable set $B \subset Y$ such that $(t \tilde{\circ} m)(B) = 0$ and $q(B) > 0$. We show that $(s \tilde{\circ} t \tilde{\circ} m)(f^{-1}(B)) = 0$ and $p(f^{-1}(B)) > 0$. On one hand, we have

$$(s \tilde{\circ} t \tilde{\circ} m)(f^{-1}(B)) = \int_B s_y(f^{-1}(B)) d(t \tilde{\circ} m) \leq (t \tilde{\circ} m)(B) = 0,$$

but on the other hand, we have $p(f^{-1}(B)) = q(B) > 0$. Therefore

$$RE((g, t) \circ (f, s)) = \infty = RE((f, s)) + RE((g, t)).$$

Case III : (f, s) is not absolutely coherent. This case can be found in the appendix. However, this proof is neither trivial nor boring.

We can thus conclude with

$$RE((g, t) \circ (f, s)) = \infty = RE((f, s)) + RE((g, t)).$$

■

We have thus shown that RE is a well-defined functor from **PolStat** to $[0, \infty]$.

4.1 Convex linearity

We show below that the relative entropy functor satisfies a convex linearity property. In [BF14] convexity looks familiar; here since we are performing “large” sums we have to express it as an integral. First we define a localized version of the relative entropy.

Definition 16. Given an arrow $(f, s) : (X, p) \rightarrow (Y, q)$ in **PolStat**. We define the *local relative entropy at y* as

$$RE((f, s)_y) := \begin{cases} \int_{f^{-1}(y)} \log \left(\frac{dp_y}{d(s \tilde{\circ} q)_y} \right) dp_y & \text{if } p_y \ll (s \tilde{\circ} q)_y \\ \infty & \text{otherwise.} \end{cases}$$

where $(f, s)_y$ is the morphism (f, s) restricted to the pair of Polish spaces $f^{-1}(y)$ and y .

Note that Lemma 22 in the appendix says that $s_y = (s \tilde{\circ} q)_y$ q -almost everywhere. Thus, there is no notational clash between the kernel s_y and the conditional probability of $(s \tilde{\circ} q)$ given y .

Theorem 17 (Convex Linearity). We have

$$RE((f, s)) = \int_Y RE((f, s)_y) dq.$$

Proof. **Case I : (f, s) is absolutely coherent.** Note that by Lemma 23, $p_y \ll (s \tilde{\circ} q)_y$ almost everywhere. So we have

$$RE((f, s)) = \int_X \log \left(\frac{dp}{d(s \tilde{\circ} q)} \right) dp \tag{14}$$

$$= \int_Y \int_{f^{-1}(y)} \log \left(\frac{dp}{d(s \tilde{\circ} q)} \right) dp_y dq \tag{15}$$

$$= \int_Y \int_{f^{-1}(y)} \log \left(\frac{dp_y}{d(s \tilde{\circ} q)_y} \right) dp_y dq \tag{16}$$

$$= \int_Y RE((f, s)_y) dq. \tag{17}$$

We get (15) by the decomposition theorem and (16) by applying Lemma 23. ■

Case II : (f, s) is not absolutely coherent. By the hypothesis of (f, s) not being absolutely coherent, there is a measurable set $A \subset X$ such that $(s \tilde{\circ} q)(A) = 0$ and $p(A) > 0$. Applying lemma 22, on one hand we have

$$\int_Y (s \tilde{\circ} q)_y(A) dq = \int_Y s_y(A) dq = (s \tilde{\circ} q)(A) = 0,$$

but on the other hand we have

$$\int_Y p_y(A) dq = p(A) > 0.$$

Hence, the subset of Y on which $p_y \ll (s \tilde{\circ} q)_y$ doesn't hold contains a set of measure strictly greater than 0. Therefore,

$$RE((f, s)) = \infty = \int_Y RE((f, s)_y) dq.$$

4.2 Lower-semicontinuity

We start with a crucial lemma which is of independent interest.

Lemma 18. Given a sequence of probability measures q_i on Y and given a sequence of kernels $s_i : Y \rightarrow \mathcal{G}(X)$, if $q_i \Rightarrow q$ and $s_i \rightarrow s$ uniformly, then $(s_i \tilde{\circ} q_i) \Rightarrow (s \tilde{\circ} q)$.

Proof. The trick is to recall that $(s_i \tilde{\circ} q_i) := \mu_X \circ \mathcal{G}(s_i) \circ q_i$ is a sequence in $\mathcal{G}(X)$ and $\mu_X : \mathcal{G}(\mathcal{G}(X)) \rightarrow \mathcal{G}(X)$ is a continuous function where we have the weak topology on the space of measures. By the continuity of μ_X , it is thus sufficient to show that the sequence $\mathcal{G}(s_i) \circ q_i$ converges to $\mathcal{G}(s) \circ q$ in $\mathcal{G}(\mathcal{G}(X))$.

By [Par67, Theorem 6.1], the sequence $\mathcal{G}(s_i) \circ q_i$ converges to $\mathcal{G}(s) \circ q$ in $\mathcal{G}(\mathcal{G}(X))$ if and only if

$$\lim_{i \rightarrow \infty} \int_{\mathcal{G}(X)} g d(\mathcal{G}(s_i) \circ q_i) = \int_{\mathcal{G}(X)} g d(\mathcal{G}(s) \circ q)$$

for all $g : \mathcal{G}(X) \rightarrow [0, \infty)$ bounded and uniformly continuous.

Let $g : \mathcal{G}(X) \rightarrow [0, \infty)$ be an arbitrary bounded and uniformly continuous function. Since, by definition we have $(\mathcal{G}(s_i) \circ q_i)(\cdot) = q_i(s_i^{-1}(\cdot))$ and $(\mathcal{G}(s) \circ q)(\cdot) = q(s^{-1}(\cdot))$, the above equality becomes

$$\lim_{i \rightarrow \infty} \int_Y (g \circ s_i) dq_i = \int_Y (g \circ s) dq.$$

By definition of weak convergence, for a fixed i , we get

$$\lim_{j \rightarrow \infty} \int_Y (g \circ s_i) dq_j = \int_Y (g \circ s_i) dq.$$

Since, $g \circ s_i$ are measurable, bounded and $g \circ s_i \rightarrow g \circ s$, for a fixed j , we can apply the *Dominated Convergence theorem* (see for example [Dud89]) to get

$$\lim_{i \rightarrow \infty} \int_Y (g \circ s_i) dq_j = \int_Y (g \circ s) dq_j.$$

Note that not only does $g \circ s_i \rightarrow g \circ s$, but since g is uniformly continuous and $s_i \rightarrow s$ uniformly, we also have $g \circ s_i \rightarrow g \circ s$ uniformly.

To conclude $\lim_{i \rightarrow \infty} \int_Y (g \circ s_i) dq_i = \int_Y (g \circ s) dq$, as is desired, it is sufficient to show that given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that, whenever $i \geq N$,

$$\left| \int_Y g \circ s_i dq_j - \int_Y g \circ s dq_j \right| \leq \epsilon, \forall j \in \mathbb{N}.$$

Let $\epsilon > 0$, since $g \circ s_i \rightarrow g \circ s$ uniformly, we can pick $N \in \mathbb{N}$ such that whenever $i \geq N$,

$$|(g \circ s_i)(y) - (g \circ s)(y)| \leq \epsilon, \forall y \in Y.$$

Then, whenever $i \geq N$ we have,

$$\left| \int_Y g \circ s_i dq_j - \int_Y g \circ s dq_j \right| \leq \int_Y |g \circ s_i - g \circ s| dq_j \leq \epsilon \cdot q_j(Y) \leq \epsilon, \forall j \in \mathbb{N}.$$

■

We now show lower-semicontinuity.

Theorem 19 (Lower semi-continuity). Given $p_i \Rightarrow p$ and a sequence of kernels $s_i \rightarrow s$ uniformly, then

$$RE \left(\begin{array}{ccc} & \xleftarrow{s} & \\ (X, p) & & (Y, q) \\ & \xrightarrow{f} & \end{array} \right) \leq \liminf_{i \rightarrow \infty} RE \left(\begin{array}{ccc} & \xleftarrow{s_i} & \\ (X, p_i) & & (Y, q_i) \\ & \xrightarrow{f} & \end{array} \right).$$

Proof. By hypothesis, we have $p_i \Rightarrow p$ and since f is measure preserving, we also have $q_i \Rightarrow q$. Applying lemma 18 to $q_i \Rightarrow q$ and $s_i \rightarrow s$ uniformly, we get $(s_i \tilde{\circ} q_i) \Rightarrow (s \tilde{\circ} q)$. We can therefore apply a theorem due to Posner, [Pos75, Theorem 1] with $p_i \Rightarrow p$ and $(s_i \tilde{\circ} q_i) \Rightarrow (s \tilde{\circ} q)$ to conclude that $RE((f, s)) \leq \liminf_{i \rightarrow \infty} RE((f, s_i))$. ■

5 Uniqueness

We now show that the relative entropy is, up to a multiplicative constant, the unique functor satisfying the conditions established so far. We use the result of Baez and Fritz as a starting point and then adapt a technique used to show that finitely supported measures are weakly dense.

We recall the main theorem of Baez and Fritz [BF14] on **FinStat**.

Theorem 20. Suppose that a functor

$$F : \mathbf{FinStat} \rightarrow [0, \infty]$$

is lower semicontinuous, convex linear and vanishes on **FP**. Then for some $0 \leq c \leq \infty$ we have $F(f, s) = cRE_{fin}(f, s)$ for all morphisms (f, s) in **FinStat**.

We extend this characterization to **PolStat**.

Theorem 21. Suppose that a functor

$$F : \mathbf{PolStat} \rightarrow [0, \infty]$$

is lower semicontinuous, convex linear and vanishes on FP . Then for some $0 \leq c \leq \infty$ we have $F(f, s) = cRE(f, s)$ for all morphisms.

Proof. Since F satisfies all the above properties on **FinStat**, we can apply the main theorem of Baez and Fritz [BF14] in order to establish that $F = cRE_{fin} = cRE$ for all morphisms in the subcategory **FinStat**. We show that F extends uniquely to cRE on all morphisms in **PolStat**.

By convexity of F , for an arbitrary morphism (f, s) from (X, p) to (Y, q) , we have

$$F((f, s)) = \int_Y F((f, s)_y) dq,$$

so F is totally described by its local relative entropies. It is thus sufficient to show $F = cRE$ on an arbitrary morphism $(f, s) : (X, p) \rightarrow (\{y\}, \delta_y)$. The crucial step to show $F((f, s)) = cRE((f, s))$ is to construct sequences of finitely supported measures p_n and s_n such that $p_n \Rightarrow p$ and $s_n \Rightarrow s$. We adapt the argument from [Par67, Theorem 6.5] in order to define p_n and s_n as follows:

Since X is separable we can, for each $n \in \mathbb{N}$, partition X into disjoint measurable sets, i.e., $X = \bigcup_j A_{nj}$ such that $A_{nj} \cap A_{ni} = \emptyset$ if $i \neq j$, A_{nj} measurable and the diameter of $A_{nj} \leq 1/n$ for all j . Let $x_{nj} \in A_{nj}$ be arbitrary. Let p_n be the measure with masses $p(A_{nj})$ at the points x_{nj} , respectively. Similarly, let s_n be the measure with masses $s(A_{nj})$ at the points x_{nj} , respectively. We denote by $\tilde{f}_n : X \rightarrow \bigcup_j \{x_{nj}\}$ the function that maps an element $x \in A_{nj}$ to the element x_{nj} and we write f_n for the function that maps every $x_{nj} \in \bigcup_j \{x_{nj}\}$ to y ; we have $f = f_n \circ \tilde{f}_n$. We denote by \tilde{s}_n , the conditional probability measure along \tilde{f}_n ; we have $s = \tilde{s}_n \circ s_n$. To summarize, we have the following construction:

$$\begin{array}{ccccc} (X, p) & \xleftarrow{\tilde{s}_n} & (\bigcup_j \{x_{nj}\}, p_n) & \xleftarrow{s_n} & (\{y\}, \delta_y) \\ & \searrow \tilde{f}_n & \searrow f_n & & \\ & & & \xRightarrow{\text{Composition}} & (X, p) \xleftarrow{s} (\{y\}, \delta_y) \end{array}$$

By the hypothesis that F is a functor, for all $n \in \mathbb{N}$, we have the following inequality

$$F((f_n, s_n)) \leq F((f, s)). \quad (18)$$

Viewing p_n and s_n as measures on X and following the argument from [Par67, Theorem 6.5], we have that $p_n \Rightarrow p$ and $s_n \Rightarrow s$. Hence, by the lower-semi continuity of F , we also have the inequality

$$F((f, s)) \leq \liminf_{n \rightarrow \infty} F((f_n, s_n)). \quad (19)$$

Since (f_n, s_n) is in **FinStat**, recall that we have $F((f_n, s_n)) = cRE((f_n, s_n))$. Thus, combining (18) and (19), we get

$$\limsup_{n \rightarrow \infty} cRE((f_n, s_n)) \leq F((f, s)) \leq \liminf_{n \rightarrow \infty} cRE((f_n, s_n)),$$

therefore $F((f, s)) = cRE((f, s))$ as desired. ■

6 Conclusions and Further Directions

As promised, we have given a categorial characterization of relative entropy on Polish spaces. This greatly broadens the scope of the original work by Baez et al. [BFL11, BF14]. However, the main motivation is to study the role of entropy arguments in machine learning. These appear in various ad-hoc ways in machine learning but with the appearance of the recent work by Danos and his co-workers [DG15, CDDG17, DDG16] we feel that we have the prospect of a mathematically well-defined framework on which to understand Bayesian inversion and its interplay with entropy. The most recent paper in this series [CDDG17] adopts a point-free approach introduced in [CDPP09, CDPP14]. It would be interesting to extend our definitions to a point-free situation. There are also many interesting questions with regard to understanding the “algebra of entropy”; see the book by Yeung [Yeu08] for a taste of these ideas.

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A Proofs

Lemma 22. Given an arrow $(f, s) : (X, p) \rightarrow (Y, q)$ in **PolStat**. Let $\{(s \tilde{\circ} q)_y\}_{y \in Y}$ denote the collection of conditional probability measures of $(s \tilde{\circ} q)$ conditioned by Y , then

$$(s \tilde{\circ} q)_y = s_y \text{ } q\text{-almost everywhere.}$$

We just have to show that $\{s_y\}_{y \in Y}$ satisfies the three properties implied by the *disintegration theorem*. We prove the third one; the first two being obvious.

3 (iii) : For every Borel-measurable function $h : X \rightarrow [0, \infty]$,

$$\int_X h d(s \tilde{\circ} q) = \int_{y \in Y} \int_{f^{-1}(y)} h ds_y dq.$$

Proof. Let's assume as a special case that h is the indicator function for a measurable set $E \subset X$. Then, we have

$$\int_X h d(s \tilde{\circ} q) = \int_E d(s \tilde{\circ} q) = (s \tilde{\circ} q)(E) = \int_{y \in Y} s_y(E) dq = \int_{y \in Y} \int_{f^{-1}(y)} h ds_y dq.$$

We have shown that it is true for any indicator function. By linearity, it is true for any simple function and then, by the Monotone Convergence Theorem, it is true for all Borel-measurable functions $h : X \rightarrow [0, \infty]$. ■

Proof of Case III of Theorem 15 By the hypothesis of (f, s) not being absolutely coherent, $p \ll s \tilde{\circ} q$ doesn't hold, so there is a measurable set $A \subset X$ such that $(s \tilde{\circ} q)(A) = 0$ and $p(A) > 0$.

We partition A into

$$A_\epsilon := \{x \in A \mid s_{f(x)}(A) > 0\} \text{ and } A_0 := \{x \in A \mid s_{f(x)}(A) = 0\}$$

and we partition Y into

$$B_\epsilon := \{y \in Y \mid s_y(A) > 0\} \text{ and } B_0 := \{y \in Y \mid s_y(A) = 0\}.$$

We argue that $(s \tilde{\circ} t \tilde{\circ} m)(A_0) = 0$ and $p(A_0) > 0$.

Since $A_0 \subset f^{-1}(B_0)$, $f^{-1}(B_\epsilon)$ is disjoint from A_0 , so for all $y \in B_\epsilon$ we have $s_y(A_0) = 0$ because their support is disjoint from A_0 . On one hand, we thus have

$$\begin{aligned}
(s \tilde{\circ} t \tilde{\circ} m)(A_0) &= \int_Y s_y(A_0) d(t \tilde{\circ} m) \\
&= \int_{B_0} s_y(A_0) d(t \tilde{\circ} m) + \int_{B_\epsilon} s_y(A_0) d(t \tilde{\circ} m) \\
&= \int_{B_0} s_y(A_0) d(t \tilde{\circ} m) \\
&\leq \int_{B_0} s_y(A) d(t \tilde{\circ} m) \\
&= 0.
\end{aligned}$$

On the other hand, since we have $p(A_0) + p(A_\epsilon) = p(A) > 0$ and $A_\epsilon \subset f^{-1}(B_\epsilon)$, it suffices to show $p(f^{-1}(B_\epsilon)) = 0$ to conclude $p(A_0) > 0$.

By hypothesis, we have

$$(s \tilde{\circ} q)(A) = \int_{B_0} s_y(A) dq + \int_{B_\epsilon} s_y(A) dq = 0,$$

so $q(B_\epsilon) = 0$ and because f is measure preserving, we have $p(f^{-1}(B_\epsilon)) = q(B_\epsilon) = 0$ as desired.

So $(g \circ f, s \tilde{\circ} t)$ is not absolutely coherent. This completes the proof of this case.

Proof of a lemma used in Theorem 17

Lemma 23. Given

$$(X, p) \xrightarrow{f} (Y, q) \xleftarrow{f} (X, p')$$

where f is a continuous function preserving the measure of both Borel probability measures p and p' . If $p \ll p'$, then

$$\frac{dp_y}{dp'_y}(x) = \frac{dp}{dp'}(x) \quad p'\text{-almost everywhere.}$$

Proof. For an arbitrary measurable function $h : X \rightarrow [0, \infty]$, we have

$$\begin{aligned}
\int_X h dp &= \int_Y \left(\int_{f^{-1}(y)} h dp_y \right) dq \\
&= \int_Y \left(\int_{f^{-1}(y)} h \frac{dp_y}{dp'_y} dp'_y \right) dq.
\end{aligned}$$

We have applied the disintegration theorem on the first line and the Radon-Nikodym theorem on the second. To show that $\frac{dp_y}{dp'_y}(x) = \frac{dp}{dp'}(x)$ p' -almost everywhere we show that we can substitute $\frac{dp_y}{dp'_y}$ for $\frac{dp}{dp'}$ in the above decomposition and get the same result.

We have

$$\begin{aligned} \int_Y \int_{f^{-1}(y)} h \frac{dp}{dp'} dp'_y dq &= \int_X h \frac{dp}{dp'} dp' \\ &= \int_X h dp, \end{aligned}$$

where on the first line we have applied the disintegration theorem on the measurable function $h \frac{dp}{dp'}$ and the Radon-Nikodym theorem on the second. ■